

Maximal symmetry at the speed of light

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Abstract

We propose a relativistic version of the cosmological principle and confront it to the Hubble diagram of supernovae and other probes.

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1 Introduction

Differential equations governing the time evolution of a field (of competitors) is one of the main paradigms in physics. To obtain a (locally) unique solution we must specify initial data of the field on a Cauchy surface (the starting line). Even for relativistic wave equations, we are used to taking the Cauchy surface space-like. In the absence of super-luminal signals, we would prefer to remain humble and use light-like ‘Cauchy’ surfaces. In other words we acknowledge several obstacles to the idealisation behind space-like Cauchy surfaces: *(i)* The starting line is too long and in the short period allocated to us since de-ionisation, we cannot observe it entirely using photons, that travel only at the speed of light. *(ii)* The competitors do not remain at rest on their marks prior to kick-off. You know how complicated a sailing regatta is at the starting line. Now add to it another complication: The jury is only able to locate the boats waiting around the starting line with go-betweens traveling at speeds comparable to the speed of the waiting boats.

We do not see how to make sense out of this complicated space-like initial value problem in cosmology without superluminal go-betweens. On the other hand, the Cauchy problem of relativistic equations with light-like surfaces is much more involved than with space-like ones [1], but this is not the subject of the present paper concerned with the Killing equation on certain light-like surfaces.

To set the stage we will give a pedestrian derivation of the Robertson-Walker metrics of negative, vanishing and positive curvatures. It is naive, but can be generalised from space-like surfaces of ‘simultaneity’ to the light-like ones of our past light-cones. The ensuing metrics together with Einstein’s equations are then confronted to the Hubble diagram of supernovae and other probes.

2 Isometry groups

Let us recall briefly the basic definitions of isometries and a few important theorems.

Consider a (pseudo) Riemannian manifold (M, g) with arbitrary signature and of dimension d . In coordinates x^μ its metric g is expressed by its metric tensor, $d\tau^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$. Let f be a diffeomorphism on M . They form a group written $Diff(M)$ which is infinite dimensional and not a Lie group. We denote the Jacobian of f in coordinates by

$$\Lambda^{\bar{\mu}}{}_{\mu}(x) := \frac{\partial f^{\bar{\mu}}}{\partial x^{\mu}}(x). \quad (1)$$

By definition f is an isometry if

$$g_{\mu\nu}(x) = \left(\Lambda^T\right)_{\mu}^{\bar{\mu}}(x) g_{\bar{\mu}\bar{\nu}}(f(x)) \Lambda^{\bar{\nu}}{}_{\nu}(x) \quad (2)$$

in all charts. The isometries form a subgroup $Iso(M, g) \subset Diff(M)$. The main theorem says that the isometry group is a Lie group of finite dimension less than or equal to $d(d+1)/2$. The (pseudo) Riemannian manifold is said to be maximally symmetric if its isometry group is of maximal dimension $d(d+1)/2$. In dimension 1+3 there are three types of maximally symmetric spaces, anti de Sitter, Minkowski and de Sitter spaces. They solve the vacuum Einstein equations with negative, vanishing and positive cosmological constant, $\Lambda = 3\sigma k^2$ in

the notations defined below. Similarly in dimensions 0+3, we have the maximally symmetric spaces: the pseudo-spheres, Euclidean \mathbb{R}^3 and the spheres.

Equation (2) is a functional differential equation and it is practical to turn it into a differential equation by considering infinitesimal diffeomorphisms. Upon linearisation $f(x) = x + \xi(x) + o(\xi^2)$, where $\xi = \xi^\alpha \partial/\partial x^\alpha$ is a vector field, equation (2) becomes the Killing equation:

$$\xi^\alpha \frac{\partial}{\partial x^\alpha} g_{\mu\nu} + \frac{\partial \xi^{\bar{\mu}}}{\partial x^{\bar{\mu}}} g_{\bar{\mu}\nu} + \frac{\partial \xi^{\bar{\nu}}}{\partial x^{\bar{\nu}}} g_{\mu\bar{\nu}} = 0. \quad (3)$$

It serves two purposes: if the generators of the isometry group are given, the Killing equation is a first order differential equation in the metric tensor. Its solutions tell us what metrics are allowed by the given symmetries. On the other hand for a given metric, the Killing equation is a first order differential equation in the vector fields. Its solutions generate the isometry group. In both cases there are global issues concerning the patching up of charts that we will happily ignore in this paper.

3 A pedestrian derivation of the Robertson-Walker metrics

In this section we recall a local derivation of the Robertson-Walker metrics using maximal symmetry on spacetime and on surfaces of ‘simultaneity’. For a thermodynamic derivation the interested reader is referred to Jean-Marie Souriau [2].

Our starting points are the maximally symmetric spacetimes: anti de Sitter $\sigma = -1$, Minkowski $\sigma = 0$ and de Sitter $\sigma = +1$ in polar coordinates (t, r, θ, φ) , where t is cosmic time and r is the geodesic ‘distance’. (It is usually denoted by χ , but we will use the letter χ later for a different coordinate.) We will use the following auxiliary functions:

$$s(r) := \begin{cases} \sinh(kr)/k & \sigma = -1 \\ r & \sigma = 0 \\ \sin(kr)/k & \sigma = +1 \end{cases}, \quad (4)$$

where k is positive and has the dimension of an inverse meter, and

$$c(r) := \begin{cases} \cosh(kr) & \sigma = -1 \\ 1 & \sigma = 0 \\ \cos(kr) & \sigma = +1 \end{cases}. \quad (5)$$

Note the continuity properties

$$\lim_{k \rightarrow 0} s_{\sigma=-1} = s_{\sigma=0}, \quad \lim_{k \rightarrow 0} s_{\sigma=+1} = s_{\sigma=0}, \quad (6)$$

$$\lim_{k \rightarrow 0} c_{\sigma=-1} = c_{\sigma=0}, \quad \lim_{k \rightarrow 0} c_{\sigma=+1} = c_{\sigma=0}. \quad (7)$$

The following relations will be useful:

$$\sigma k^2 s^2 + c^2 = 1, \quad s' = c, \quad c' = -\sigma k^2 s, \quad (c/s)' = -1/s^2, \quad (s/c)' = 1/c^2. \quad (8)$$

Likewise we define:

$$\bar{s}(t) := \begin{cases} \sin(kt)/k & \sigma = -1 \\ t & \sigma = 0 \\ \sinh(kt)/k & \sigma = +1 \end{cases}, \quad \bar{c}(t) := \begin{cases} \cos(kt) & \sigma = -1 \\ 1 & \sigma = 0 \\ \cosh(kt) & \sigma = +1 \end{cases}, \quad (9)$$

with similar relations. Now we can write down the metrics of our maximally symmetric spacetimes:

$$d\tau^2 = dt^2 - \bar{c}^2(t) \left[dr^2 + s^2(r) d\theta^2 + s^2(r) \sin^2 \theta d\varphi^2 \right] \quad (10)$$

with the ten Killing vectors, three rotations, three space ‘translations’, three Lorentz boosts, and a time ‘translation’:

$$\begin{aligned} R_x &= -\sin \varphi \frac{\partial}{\partial \theta} - \frac{\cos \theta}{\sin \theta} \cos \varphi \frac{\partial}{\partial \varphi}, & T_x &= \sin \theta \cos \varphi \frac{\partial}{\partial r} + \frac{c}{s} \cos \theta \cos \varphi \frac{\partial}{\partial \theta} - \frac{c}{s} \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ R_y &= +\cos \varphi \frac{\partial}{\partial \theta} - \frac{\cos \theta}{\sin \theta} \sin \varphi \frac{\partial}{\partial \varphi}, & T_y &= \sin \theta \sin \varphi \frac{\partial}{\partial r} + \frac{c}{s} \cos \theta \sin \varphi \frac{\partial}{\partial \theta} + \frac{c}{s} \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ R_z &= \frac{\partial}{\partial \varphi}, & T_z &= \cos \theta \cdot \frac{\partial}{\partial r} - \frac{c}{s} \sin \theta \cdot \frac{\partial}{\partial \theta}, \\ L_x &= s \sin \theta \cos \varphi \frac{\partial}{\partial t} + \frac{\bar{s}}{c} c \sin \theta \cos \varphi \frac{\partial}{\partial r} + \frac{\bar{s}}{c} \frac{1}{s} \cos \theta \cos \varphi \frac{\partial}{\partial \theta} - \frac{\bar{s}}{c} \frac{1}{s} \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ L_y &= s \sin \theta \sin \varphi \frac{\partial}{\partial t} + \frac{\bar{s}}{c} c \sin \theta \sin \varphi \frac{\partial}{\partial r} + \frac{\bar{s}}{c} \frac{1}{s} \cos \theta \sin \varphi \frac{\partial}{\partial \theta} + \frac{\bar{s}}{c} \frac{1}{s} \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ L_z &= s \cos \theta \cdot \frac{\partial}{\partial t} + \frac{\bar{s}}{c} c \cos \theta \cdot \frac{\partial}{\partial r} - \frac{\bar{s}}{c} \frac{1}{s} \sin \theta \cdot \frac{\partial}{\partial \theta}, \\ T_t &= c \frac{\partial}{\partial t} - \sigma k^2 \frac{\bar{s}}{c} s \frac{\partial}{\partial r}. \end{aligned} \quad (11)$$

As a check, let us compute a few Lie brackets:

$$\begin{aligned} [R_x, R_y] &= -R_z, & [R_x, T_z] &= T_y, & [T_y, T_z] &= -\sigma k^2 R_x, & [R_z, T_z] &= 0, \\ [L_y, L_z] &= R_x, & [R_x, L_z] &= L_y, & [R_z, L_z] &= 0, & [T_x, L_z] &= 0, \\ [T_z, L_z] &= T_t, & [R_z, T_t] &= 0, & [T_z, T_t] &= -\sigma k^2 L_z, & [L_z, T_t] &= -T_z. \end{aligned} \quad (12)$$

These Killing vectors generate the identity components of the isometry groups, the anti de Sitter group $O(2, 3)$ for $\sigma = -1$, the Poincaré group $O(1, 3) \ltimes \mathbb{R}^4$ for $\sigma = 0$ and the de Sitter group $O(1, 4)$ for $\sigma = 1$.

Maximally symmetric spacetimes are too rigid to match our observations of the universe. The conventional way to relax the symmetry is to only postulate maximal symmetry on space-like 3-surfaces of ‘simultaneity’, $t = t_0$. These are only invariant under the three rotations R_x, R_y, R_z and the three ‘translations’ T_x, T_y, T_z generating the identity components of the Lorentz group $O(1, 3)$ for $\sigma = -1$, of the 3-dimensional Euclidean group $O(3) \ltimes \mathbb{R}^3$ for $\sigma = 0$ and of $O(4)$ for $\sigma = 1$. They are the isometry groups of pseudo-spheres, the Euclidean 3-space and spheres of radius $\bar{c}(t_0)/k$. Note a dangerous trap with pseudo-spheres. Their isometry groups are isomorphic to the Lorentz group. Its elements are genuine rotations, but the Lorentz transformations are fake, for there is no time. They are fake translations, fake because they do not commute.

The most general spacetime metrics solving the Killing equations (3) for the six Killing vectors $R_x, R_y, R_z, T_x, T_y, T_z$ are the Robertson-Walker metrics

$$d\tau^2 = b^2(t) dt^2 - a^2(t) \left[dr^2 + s^2(r) d\theta^2 + s^2(r) \sin^2 \theta d\varphi^2 \right], \quad (13)$$

with positive functions $a(t)$ and $b(t)$. By a transformation of the time coordinate we may achieve $b \equiv 1$, in ‘cosmic time’, that we still denote by t , or we may achieve $b \equiv a$, in ‘conformal time’, that we denote by η . The link between cosmic time and conformal time is given by $d\eta/dt = 1/a(t)$. The Robertson-Walker spacetimes are foliated by the time coordinate with 3-dimensional, space-like leaves of maximal symmetry, pseudo-spheres, Euclidean 3-space and spheres.

Note that in all three cases the time axis is automatically ‘perpendicular’ to the leaves without invoking a ‘Weyl principle’. This comes from the embedding of the six 3-dimensional Killing vectors (on the leaves) into spacetime, an embedding that these vectors inherit from the maximally symmetric spacetimes we started from. For a recent analysis of ‘Weyl’s principle’ and an example relaxing it, see Marinoni & Steigerwald [3].

4 Light-like leaves

Our task is to exhibit a spacetime foliated by 3-dimensional, light-like leaves, our past light-cones, with ‘maximal symmetry’. We use the quotation marks because the main theorem above fails for general degenerate metrics: They may well have infinite dimensional isometry groups. Take the direct product of the real line with vanishing ‘metric’ and the round 2-sphere. Its isometry group is $\text{Diff}(\mathbb{R}) \times O(3)$. We find it encouraging that this problem will not appear when we repeat the above steps for the three families of maximally symmetric spacetimes (10), replacing however the space-like leaves of ‘simultaneity’ by the light-like leaves of our past light-cones.

Of course we will switch to light-cone coordinates in conformal time:

$$\begin{aligned} u &:= \frac{1}{\sqrt{2}}(\eta + r), \quad \eta = \frac{1}{\sqrt{2}}(u + v), \quad \frac{\partial}{\partial t} = \frac{1}{\sqrt{2}} \frac{1}{a} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right), \\ v &:= \frac{1}{\sqrt{2}}(\eta - r), \quad r = \frac{1}{\sqrt{2}}(u - v), \quad \frac{\partial}{\partial r} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right). \end{aligned} \quad (14)$$

In these coordinates the maximally symmetric metrics (10), $a(t) = \bar{c}(t)$, read:

$$d\tau^2 = \bar{c}^2(t(\frac{1}{\sqrt{2}}(u + v))) \left[2 du dv - s^2(\frac{1}{\sqrt{2}}(u - v)) d\theta^2 - s^2(\frac{1}{\sqrt{2}}(u - v)) \sin^2 \theta d\varphi^2 \right], \quad (15)$$

and their Lorentz transformations are:

$$\begin{aligned} L_x &= \frac{1}{\sqrt{2}} \frac{1}{\bar{c}} (s + \bar{s}c) \sin \theta \cos \varphi \frac{\partial}{\partial u} + \frac{1}{\sqrt{2}} \frac{1}{\bar{c}} (s - \bar{s}c) \sin \theta \cos \varphi \frac{\partial}{\partial v} \\ &\quad + \frac{\bar{s}}{\bar{c}} \frac{1}{s} \cos \theta \cos \varphi \frac{\partial}{\partial \theta} - \frac{\bar{s}}{\bar{c}} \frac{1}{s} \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ L_y &= \frac{1}{\sqrt{2}} \frac{1}{\bar{c}} (s + \bar{s}c) \sin \theta \sin \varphi \frac{\partial}{\partial u} + \frac{1}{\sqrt{2}} \frac{1}{\bar{c}} (s - \bar{s}c) \sin \theta \sin \varphi \frac{\partial}{\partial v} \\ &\quad + \frac{\bar{s}}{\bar{c}} \frac{1}{s} \cos \theta \sin \varphi \frac{\partial}{\partial \theta} + \frac{\bar{s}}{\bar{c}} \frac{1}{s} \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ L_z &= \frac{1}{\sqrt{2}} \frac{1}{\bar{c}} (s + \bar{s}c) \cos \theta \cdot \frac{\partial}{\partial u} + \frac{1}{\sqrt{2}} \frac{1}{\bar{c}} (s - \bar{s}c) \cos \theta \cdot \frac{\partial}{\partial v} \\ &\quad - \frac{\bar{s}}{\bar{c}} \frac{1}{s} \sin \theta \cdot \frac{\partial}{\partial \theta}, \end{aligned} \quad (16)$$

where it is understood that the arguments t and r are replaced by $t = t(\eta)$, $\eta = \frac{1}{\sqrt{2}}(u + v)$, $r = \frac{1}{\sqrt{2}}(u - v)$. On the past light-cones $u = u_0$, the 4-metrics reduce to the *degenerate* 3-metrics:

$$d\nu^2 = -\tilde{a}^2(v) \left[d\theta^2 + \sin^2 \theta d\varphi^2 \right], \quad \tilde{a}(v) := \bar{c}(t(\frac{1}{\sqrt{2}}(u_0 + v))) s(\frac{1}{\sqrt{2}}(u_0 - v)). \quad (17)$$

To compute its Killing vectors we need the derivative of \tilde{a} :

$$\tilde{a}' = \frac{d\tilde{a}(v)}{dv} = \frac{1}{\sqrt{2}} \bar{c}(t) \left[\sigma k^2 \bar{s}(t) s(r) - c(r) \right], \quad t = t(\eta), \quad \eta = \frac{1}{\sqrt{2}}(u_0 + v), \quad r = \frac{1}{\sqrt{2}}(u_0 - v), \quad (18)$$

and we find that the Killing vectors form a 6-dimensional Lie algebra spanned by the three rotations R_x, R_y, R_z and the three Lorentz transformations $\tilde{L}_x, \tilde{L}_y, \tilde{L}_z$:

$$\begin{aligned} \tilde{L}_x &= -\frac{\tilde{a}}{\tilde{a}'} \sin \theta \cos \varphi \frac{\partial}{\partial v} - \cos \theta \cos \varphi \frac{\partial}{\partial \theta} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ \tilde{L}_y &= -\frac{\tilde{a}}{\tilde{a}'} \sin \theta \sin \varphi \frac{\partial}{\partial v} - \cos \theta \sin \varphi \frac{\partial}{\partial \theta} - \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \varphi}, \\ \tilde{L}_z &= -\frac{\tilde{a}}{\tilde{a}'} \cos \theta \quad \cdot \quad \frac{\partial}{\partial v} + \sin \theta \quad \cdot \quad \frac{\partial}{\partial \theta}. \end{aligned} \quad (19)$$

Indeed, we get the same commutation relations as in (12), $[\tilde{L}_y, \tilde{L}_z] = R_x$, $[R_x, \tilde{L}_z] = \tilde{L}_y$, $[R_z, \tilde{L}_z] = 0$, and the Killing vectors generate the identity component of the Lorentz group $O(1, 3)$. These commutation relations continue to hold for arbitrary functions $\tilde{a}(v)$ with nowhere vanishing derivative, not necessarily of the form in equation (17). It is worth noting that while in the case of surfaces of ‘simultaneity’ the isometry groups are different for the three families of metrics, $\sigma = -1, 0, +1$, these three groups are isomorphic in the case of light-cones. This is remarkable because we are talking about light-cones in spacetime, not in tangent space, and they are cones only for $\sigma = 0$. For $\sigma = 1$ the past light-cones even have two singularities. Note also that this time the Lorentz transformations are not fake, they are genuine boosts. In the case $\sigma = 0$, the Lorentz boosts with tildes in equations (19) are the restrictions of the Lorentz boosts in equations (16) to the light-cone $u = u_0$ after the shift $u \rightarrow u - u_0$ and $v \rightarrow v - u_0$.

Now that we know the isometry group on the leaves, we compute – as in the last section – the most general metric on the foliated spacetimes compatible with the Killing vectors, $R_x, R_y, R_z, L_x, L_y, L_z$.

To this end it is convenient to work in still another coordinate system:

$$\chi := \sqrt{\bar{s}^2(t) - \bar{c}^2(t) s^2(r)} = \sqrt{\bar{s}^2(t) c^2(r) - s^2(r)}, \quad \psi := \text{Ar tanh} \frac{\bar{c}(t) s(r)}{\bar{s}(t)}, \quad (20)$$

with Jacobian

$$J := \begin{pmatrix} \partial \chi / \partial t & \partial \psi / \partial t \\ \partial \chi / \partial r & \partial \psi / \partial r \end{pmatrix} = \begin{pmatrix} \bar{s} \bar{c} c^2 / \chi & -s / \chi^2 \\ -\bar{c}^2 s c / \chi & \bar{s} \bar{c} c / \chi^2 \end{pmatrix} \quad \text{and} \quad J^{-1} = \begin{pmatrix} \bar{s} / \bar{c} / \chi & -s / (\bar{c}^2 c \chi) \\ s & \bar{s} c / \bar{c} \end{pmatrix}. \quad (21)$$

Then with

$$\frac{\partial}{\partial \psi} = \frac{\partial t}{\partial \psi} \frac{\partial}{\partial t} + \frac{\partial r}{\partial \psi} \frac{\partial}{\partial r} = s \frac{\partial}{\partial t} + \frac{\bar{s} c}{\bar{c}} \frac{\partial}{\partial r} \quad (22)$$

we get

$$L_z = \cos \theta \frac{\partial}{\partial \psi} - \coth \psi \sin \theta \frac{\partial}{\partial \theta}. \quad (23)$$

We find it amazing that in these coordinates the boosts are independent of σ .

It is well-known that the most general solution of the Killing equation with respect to the rotations R_x, R_y, R_z is

$$d\tau^2 = B d\chi^2 + 2S d\chi d\psi - A d\psi^2 - C d\theta^2 - C \sin^2 \theta d\varphi^2, \quad (24)$$

where A, B, C and S are functions of χ and ψ . Since all boosts can be obtained by commuting rotations with L_z , the functions A, B, C and S are determined from the Killing equation for L_z alone. This calculation is straight-forward and yields the Robertson-Walker metrics with $\sigma = -1$: $S = 0, C = a^2(\chi) \sinh^2 \psi, B = b^2(\chi)$ and $A = a^2(\chi)$.

The coordinate transformation (20) has a singularity at $\chi = 0$. To be sure that this singularity does not spoil our conclusion, we rewrite the obtained metric,

$$d\tau^2 = b^2(\chi) d\chi^2 - a^2(\chi) d\psi^2 - a^2(\chi) \sinh^2 \psi d\theta^2 - a^2(\chi) \sinh^2 \psi \sin^2 \theta d\varphi^2, \quad (25)$$

in the light-cone coordinates u and v . To simplify we choose the embedding into Minkowski space and we choose χ such that $a^2 = 2uv b^2$. Then we get

$$d\tau^2 = b^2(\sqrt{2uv}) \left[2 du dv - \frac{1}{2}(u - v)^2 d\theta^2 - \frac{1}{2}(u - v)^2 \sin^2 \theta d\varphi^2 \right]. \quad (26)$$

This metric has light-like leaves $u = u_0$ with maximal symmetry. The isometry groups are the Lorentz group (at least its connected component) as long as $d/dv [b(\sqrt{2u_0 v}) (u_0 - v)] \neq 0$.

Our little calculation indicates that the pseudo-spheres are universal with respect to the relativistic version of the cosmological principle. But we cannot exclude that there are other metrics admitting a foliation with maximally symmetric, light-like leaves; metrics that cannot be obtained from maximally symmetric spacetimes. Any such metric, of course, would be extremely interesting to be tested against cosmological observations.

Let us state this universality of pseudo-spheres differently. The Robertson-Walker metrics with $\sigma = -1$ admit two foliations: The first, $\psi = \psi_0$, has space-like leaves with maximal symmetry and the fake Lorentz group. The second, $u = u_0$, has light-like leaves with maximal symmetry and the genuine Lorentz group. We conjecture that the other two families, with spheres and planes, do not admit the second type of foliation. Before mathematicians prove or disproves this conjecture, let us see what cosmological observations tell us. Of course we start with the cleanest (because coordinate independent) test, the Hubble diagram.

5 Data analysis

We have seen above that the metrics associated with maximal symmetry on past light-cones are the Robertson-Walker metrics with $\sigma = -1$. Consequently, the Friedmann equations can be used with the restriction of a positive curvature density parameter Ω_k and we use the published results on supernovae directly to test the value of Ω_k .

The most up-to-date results for supernovae are coming from the SCP compilation experiment [4] and the latest SNLS 3 years data [5].

Figures 1 show the confidence level contours in the Ω_m, Ω_Λ plane for these two sets of supernovae. Both samples using supernovae alone seem to favour a positive value of Ω_k (i.e. $\Omega_t := \Omega_m + \Omega_\Lambda > 1$) However these two results are statistically compatible with a flat universe because of the high degeneracy between Ω_m and Ω_Λ and no conclusion can be drawn.

Future supernova experiments, like EUCLID [6], WFIRST [7] or LSST [8] won't be able to overcome this degeneracy. Using our own simulation [9] of a WFIRST like survey (2000

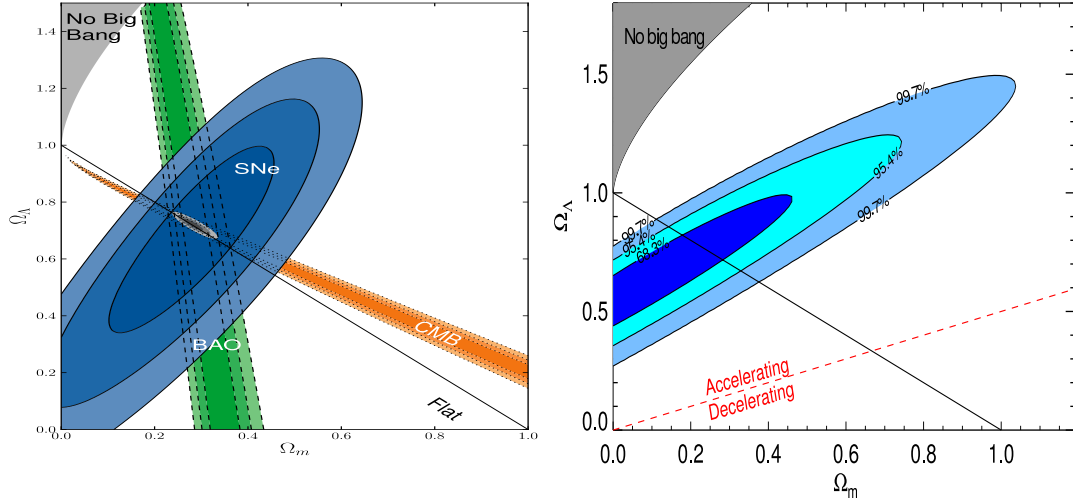


Figure 1: 68.3%, 94.4% and 99.7% confidence level contour in the $(\Omega_m, \Omega_\Lambda)$ plane for the SCP sample [4] (left hand side) and SNLS 3 years sample [5] (right hand side). Figures are taken from the references.

supernovae up to a redshift of 1.7 and intrinsic dispersion of 0.1) we find an accuracy on Ω_k of about 0.05 (statistical error only). In principle the LSST survey can do better. Using again our simulation program with 50000 supernovae per year up to a redshift of 0.8 with the same intrinsic dispersion gives a statistical accuracy of 0.02 (0.007 for 10 years running). Those results are only indicative, because – as is well known – supernova errors are limited by systematic effects like evolution with redshift, dust or intrinsic variability.

To overcome this degeneracy we can combine supernovae with other probes, like CMB, BAO and weak lensing. Again we can take these combinations directly from published results because the metric is the same and perturbation theory can be used as is. Table 1 shows the most up-to-date results on Ω_k . All results are in good agreement with each other and are compatible with a flat universe. However a positive curvature is possible at a 2σ level [4] if time evolution of dark energy equation of state is included in the fitting procedure.

Future experiments like PLANCK [10] combined with WFMOS [11] or SKA [12] would achieve an accuracy on Ω_k between $0.5 \cdot 10^{-3}$ and $2 \cdot 10^{-3}$ with a constant dark energy equation of state parameter [13]. This accuracy is degraded by a factor 10 in case of evolving dark energy equation of state [14].

A positive curvature parameter measurement Ω_k above 10^{-2} to 10^{-3} will reinforce our hypothesis of a maximally symmetric universe on our past light-cones. On the contrary a negative curvature measurement will rull out this hypothesis.

6 Epilogue

6.1 Mathematical questions

From the start, we took our light-cones embedded in maximally symmetric spacetimes. This was convenient for the calculation because it by-passed a general theory of isometry groups of

Experiments	Errors	Level	Reference
WMAP7+SN1a+BAO	$-0.0133 < \Omega_k < 0.0084$	95% CL	[15]
Union2+WMAP7+BAO	$\Omega_k = -0.004 \pm 0.007$	1σ	[16]
SNLS+WMAP7+BAO	$\Omega_k = -0.002 \pm 0.006$	1σ	[5]
SNe+BAO+CMB+ H_0	$\Omega_k = 0.002 \pm 0.005$	1σ	[4]
SNe+BAO+CMB+ H_0	$\Omega_k = 0.027^{+0.012}_{-0.011}$	1σ	[4]

Table 1: Present constraints on spatial curvature parameter. The last line includes time evolution of dark energy equation of state in the fitting procedure.

degenerate ‘metrics’, which, to the best of our knowledge, is still missing.

As already mentioned, a classification of space-times admitting foliations with maximally symmetric light-like leaves is needed in order to decide if Robertson-Walker metrics with $\sigma = 0$ or $+1$ are indeed incompatible with the relativistic cosmological principle. If we are lucky, this classification might also exhibit a new spacetime metric to be tested in cosmology. (A classification of space-times admitting foliations with maximally symmetric space-like leaves is also welcome, as it would clarify ‘Weyl’s principle’.)

Our calculation was local and we do not know under which global conditions our Killing vectors do exponentiate. Many cosmological models have space-like leaves that are maximally symmetric only locally. However for pseudo-spheres the theory of space forms is more involved than for planes and spheres [17].

6.2 Conclusions

Any change of paradigm comes with doubts and challenges. We are not sure that the tests involving perturbations and Boltzmann’s equation yield the same results for the two different foliations, the one with space-like and the one with light-like leaves. But we do hope that the light-cone coordinates yielding maximally symmetric leaves on the Robertson-Walker metrics with negative curvature can contribute to a simplified description of cosmological observations.

From the strategic point of view, we retain two pleasant features of the relativistic version of the cosmological principle: two expected road blocks did not happen. *(i)* The isometry group of the light-cone came out finite dimensional in our, admittedly pedestrian, approach. *(ii)* To get rid of simultaneity, we started out with another heresy, that of a privileged observer, us. But at the end privilege and heresy vanished.

From the physical point of view, our conclusion amounts to two disappointments: *(i)* Of course we had hoped that our relativistic version of the cosmological principle lead to metrics different from the well studied Robertson-Walker ones. This is not true. But at least there is a constraint, that of negative space-curvature. *(ii)* The second disappointment stems from the age of the authors. No statistically significant, dedicated test of the constraint will become available within our life time. Nevertheless it is not quite excluded that the Hubble diagram might show a statistically significant non-monotonicity in our future. This would rule out the relativistic version of the cosmological principle even without assuming Einstein’s equation [18].

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